

# ON EXTENDING CALIBRATION PAIRS

YONGSHENG ZHANG

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## 1. INTRODUCTION

A calibration on a Riemannian manifold  $(X, g)$  is a closed differential  $m$ -form  $\phi$  whose value at every point on every unit  $m$ -plane is at most one. The fundamental theorem of calibrated geometry in [HL82a] asserts that an  $m$ -dimensional oriented compact submanifold  $M$  (or more generally a current) for which  $\phi$  has value one a.e. on every unit tangent plane is mass-minimizing in its homology class of normal currents. We call  $(\phi, g)$  a calibration pair of  $M$  on  $X$ .

In this paper we shall create such balanced pairs for objects in various situations. The idea is to have a local calibration pair and extend it to a global one. Based on types of objects to deal with, the paper divides into two parts: the smooth case and the singular case.

Given a homologically nontrivial, oriented, connected, compact submanifold  $M$ , we show that one can conformally change a priori metric such that  $M$  becomes homologically mass-minimizing. Our existence result in every conformal class of metrics generalizes the existence theorem in [Tas93]. In his paper, Tasaki first applied a functional analysis argument of Sullivan [Sul76] for a global form  $\phi$  which has positive values on the oriented tangent planes of  $M$ ,

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and then he made use of two powerful results (Lemmas 4.1 and 4.2 in our §4.1) by Harvey and Lawson [HL82b] to build a metric  $g$  adapted to  $\phi$  so that  $(\phi, g)$  forms a calibration pair of  $M$ . Although our improvement reflects in the metric slot, the essential difference comes from the construction of calibrations. Our recipe is this. Lemma 3.4 provides a well-behaved local calibration pair. We first extend the form to a global one in §3.3. Then glue metrics accordingly in §3.4 for a global calibration pair. The case of a constellation of mutually disjoint submanifolds possibly of different dimensions is also studied in §3.5.

By Federer and Fleming [FF60] there exists at least one mass-minimizing normal current in every real-valued homology class of a compact Riemannian manifold. However the regularity of these mass-minimizing currents and their distributions are quite complicated in general. We construct nice metrics in Theorem 3.19 so that (as functionals over smooth forms) all homologically mass-minimizing currents of codimension at least 3 are just linear combinations (of integrations) over submanifolds. The thought is the following. For each dimension, the homology space has a basis that can be represented by oriented connected compact submanifolds. One can arrange these representatives so that all intersections among them are transversal. Then enough calibrations can be made to feed our need for codimensions no less than 3.

Except in low dimensions, mass-minimizing and even calibrated currents can have singularities. N. Smale [Sma99] gave the first examples of homologically mass-minimizing compact hypersurfaces with isolated singular points. In the second part of our paper, a different method for getting such creatures through calibrations is gained. We first establish an extension result Theorem 4.6 (also see Example 1) which allows us to extend a “nice” local calibration pair<sup>1</sup> of a singular submanifold around its singular set to a calibration pair on some neighborhood of the submanifold. Under certain condition a further extension to a global pair can be made. Then in Example 2 we illustrate how to build up examples satisfying the requirements in Theorem 4.6. They provide lots of instances similar to N. Smale’s.

Our local models of singularities with nice calibration pairs include all homogeneous mass-minimizing hypercones which have (coflat) calibrations singular only at the origin (see [Zhaa]), and all special Lagrangian cones (see [Joy08], [McI03], [CM04], [Has04], [HK07], [HK08], [HK12] and etc. for the diversity) that enjoy smooth calibrations. In fact, based on beautiful (but non-coflat) calibrations in [HS85] and [Law91] and further analysis, we show in [Zhab] that every area-minimizing hypercone and every oriented area-minimizing cone obtained in [Law91] can be realized as a tangent cone at a singular point of some homologically area-minimizing singular compact submanifold.

A very interesting phenomenon, that we observe in Example 3, is the existence of homologically mass-minimizing smooth submanifolds which cannot be calibrated by any smooth calibration. Actually, all coflat calibrations of the submanifold share at least one common singular point. By Remark 4.12 there are examples for which calibrations share more complicated singular sets.

Through blowing-up we get Example 4 which relates to twisted calibrations [Mur91] and integral currents mod 2 [Zie62]. It gives us a non-orientable compact singular hypersurface that is mass-minimizing in its homology class of integral currents mod 2.

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<sup>1</sup>In this case the calibration form may be singular somewhere. See the definition of coflat calibration in §2.

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## 2. PRELIMINARIES

We review some fundamental concepts and results in calibrated geometry. Readers are referred to [HL82a] for a further understanding on this subject and to [Mor08] for a quick overview of geometric measure theory.

**Definition 2.1.** Let  $\phi$  be a smooth  $m$ -form on a Riemannian manifold  $(X, g)$ . At a point  $x \in X$  we define the **comass** of  $\phi_x$  to be

$$\|\phi\|_{x,g}^* = \max \{ \phi_x(\vec{V}_x) : \vec{V}_x \text{ is a unit simple } m\text{-vector at } x \}.$$

Here “simple” means  $\vec{V}_x = e_1 \wedge e_2 \cdots \wedge e_m$  for some  $e_i \in T_x X$ .

**Remark 2.2.**  $\|\phi\|_g^*$  will be viewed as a pointwise function in this paper. In general it is merely continuous. At a point  $x$  where  $\phi_x \neq 0$ ,

$$\begin{aligned} \|\phi\|_{x,g}^* &= \max \{ \phi(\vec{V}_x) : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \|\vec{V}_x\|_g = 1 \} \\ &= \max \{ 1 / \|\vec{V}_x\|_g : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \phi(\vec{V}_x) = 1 \} \\ &= 1 / \min \{ \|\vec{V}_x\|_g : \vec{V}_x \text{ is a simple } m\text{-vector at } x \text{ with } \phi(\vec{V}_x) = 1 \}. \end{aligned}$$

**Definition 2.3.** Denote the dual complex of the de Rham complex of  $X$  by  $(\mathcal{E}'_*(X), d)$ . Elements of  $\mathcal{E}'_k(X)$  are  $k$ -dimensional de Rham **currents** (with compact support) and  $d$  is the adjoint of exterior differentiation.

**Definition 2.4.** In  $(X, g)$ , the **mass**  $\mathbf{M}(T)$  of  $T \in \mathcal{E}'_k(X)$  is defined to be

$$\sup \{ T(\psi) : \psi \text{ smooth } m\text{-form with } \sup_X \|\psi\|_g^* \leq 1 \}.$$

When  $\mathbf{M}(T) < \infty$ ,  $T$  determines a unique Radon measure  $\|T\|$  characterized by

$$\int_X f \cdot d\|T\| = \sup \{ T(\psi) : \|\psi\|_{x,g}^* \leq f(x) \}$$

for any nonnegative continuous function  $f$  on  $X$ . Therefore  $\mathbf{M}(T) = \|T\|(X)$ . Moreover, the Radon-Nikodym Theorem asserts the existence of a  $\|T\|$  measurable tangent  $m$ -vector field  $\vec{T}$  a.e. with vectors  $\vec{T}_x \in \Lambda^m T_x X$  of unit length in the dual norm of the comass norm, satisfying

$$(2.1) \quad T(\psi) = \int_X \psi_x(\vec{T}_x) d\|T\|(x) \text{ for any smooth } m\text{-form } \psi,$$

or briefly  $T = \vec{T} \cdot \|T\|$  a.e.  $\|T\|$ . When  $T$  has local finite mass, one can get Radon measure  $\|T\|$  and decomposition (2.1) as well.

**Definition 2.5.** For a function  $f$ , set  $\mathbf{spt}(f)$  to be its support. For a current  $T$ , let  $U_T$  stand for the largest open set with  $\|T\|(U_T) = 0$ . Then the support of  $T$  is denoted by  $\mathbf{spt}(T) = U_T^c$ .

**Definition 2.6.** Let  $\mathbb{M}_k(X) = \{T \in \mathcal{E}'_k(X) : \mathbf{M}(T) < \infty\}$ . Then  $N_k(X) = \{T \in \mathbb{M}_k(X) : dT \in \mathbb{M}_{k-1}(X)\}$  is the space of  $k$ -dimensional **normal currents**.

**Remark 2.7.** We view a current in  $\mathbb{M}_k$  as a functional over smooth  $k$ -form not a specific representative of generalized distribution.

Note that  $(N_*(X), d)$  form a chain complex. Recalling the natural isomorphisms established by de Rham, Federer and Fleming:

$$H_*(\mathcal{E}'_*(X)) \cong H_*(X; \mathbb{R}) \cong H_*(N_*(X))$$

we identify these three homology groups.

**Definition 2.8.** A smooth form  $\phi$  on  $(X, g)$  is called a **calibration** if  $\sup_X \|\phi\|_g^* = 1$  and  $d\phi = 0$ . Such a triple  $(X, \phi, g)$  is called a **calibrated manifold**. If  $M$  is an oriented submanifold with  $\phi|_M$  equal to the volume form of  $M$ , then  $(\phi, g)$  is a **calibrated pair** of  $M$  on  $X$ . We say  $\phi$  **calibrates**  $M$  and  $M$  **can be calibrated** in  $(X, g)$ .

**Definition 2.9.** Let  $\phi$  be a calibration on  $(X, g)$ . We say that a current  $T$  of local finite mass is **calibrated** by  $\phi$ , if  $\phi_x(\vec{T}_x) = 1$  a.a.  $x \in X$  for  $\|T\|$ .

**Remark 2.10.** For an oriented compact submanifold  $M$ , the current  $[[M]] = \int_M \cdot$  is calibrated if and only if  $M$  is calibrated.

The following is the fundamental theorem of calibrated geometry in [HL82a].

**Theorem 2.11.** If  $T$  is a calibrated current<sup>2</sup> with compact support in  $(X, \phi, g)$  and  $T'$  is any compactly supported current homologous to  $T$  (i.e.,  $T - T'$  is a boundary and in particular  $dT = dT'$ ), then

$$\mathbf{M}(T) \leq \mathbf{M}(T')$$

with equality if and only if  $T'$  is calibrated as well.

It is often useful to allow calibrations to have certain singularities.

**Definition 2.12.** Let  $\phi$  be a calibration of degree  $m$  on  $X - S_\phi$ , where  $S_\phi$  is a closed subset of  $X$  of Hausdorff  $m$ -measure zero. Then  $\phi$  is called a **coflat calibration** on  $X$ . We say  $\phi$  **calibrates** a current, if it is calibrated by  $\phi$  on  $X - S_\phi$ .

**Remark 2.13.** Actually there is a coflat version (Theorem 4.9 in [HL82a]) of the fundamental theorem of calibrated geometry, and a current calibrated by a coflat calibration is homologically mass-minimizing as well.

### 3. SMOOTH CASE

We shall use some properties of comass. Especially, Lemma 3.4 is crucial to our methods and Lemma 3.3 provides certain control on comass while gluing metrics.

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<sup>2</sup>It is called a positive  $\phi$ -current in [HL82a].

### 3.1. Properties of comass.

**Lemma 3.1.** *For any metric  $g$ ,  $m$ -form  $\phi$  and positive function  $f$  on  $X$ ,*

$$\|\phi\|_{f \cdot g}^* = f^{-\frac{m}{2}} \cdot \|\phi\|_g^*.$$

**Proof.** By the formula in Remark 2.2. ■

**Lemma 3.2.** *For any  $m$ -form  $\phi$  and metrics  $g' \geq g$  on  $X$ , we have*

$$\|\phi\|_{g'}^* \leq \|\phi\|_g^*.$$

**Proof.** By the definition of comass. ■

**Lemma 3.3** (Comass control for gluing procedure). *For any  $m$ -form  $\phi$ , positive functions  $a$  and  $b$ , and metrics  $g_1$  and  $g_2$ , it follows*

$$(3.1) \quad \|\phi\|_{ag_1+bg_2}^* \leq \frac{1}{\sqrt{a^m \cdot \frac{1}{\|\phi\|_{g_1}^{*2}} + b^m \cdot \frac{1}{\|\phi\|_{g_2}^{*2}}}}$$

where  $\frac{1}{0}$  and  $\frac{1}{+\infty}$  are identified with  $+\infty$  and  $0$  respectively.

**Proof.** The statement is trivial at where  $\phi$  vanishes. Suppose  $\phi_x \neq 0$  at a point  $x$ . In the subspace spanned by a simple  $m$ -vector  $\vec{V}_x$ , there exists an orthonormal basis  $(e_1, \dots, e_m)$  of  $g_1$ , under which  $g_2$  is diagonalized as  $\text{diag}(\lambda_1, \dots, \lambda_m)$  for some  $\lambda_i > 0$ . Let  $\vec{V}_x = te_1 \wedge \dots \wedge e_m$ , then

$$(3.2) \quad \begin{aligned} \|\vec{V}_x\|_{ag_1+bg_2}^2 &= t^2(a + b\lambda_1) \cdots (a + b\lambda_m) \\ &= t^2[a^m + \cdots + b^m \prod \lambda_i] \\ &\geq t^2 a^m + t^2 b^m \prod \lambda_i \\ &= a^m \|\vec{V}_x\|_{g_1}^2 + b^m \|\vec{V}_x\|_{g_2}^2. \end{aligned}$$

By Remark 2.2, (3.2) implies (3.1). ■

**Lemma 3.4** (Comass one lemma). *Suppose  $(E, \pi)$  is a disk bundle over  $M$  (as the zero section) and  $g$  is a Riemannian metric on  $E$ . Then each fiber is perpendicular to  $M$  if and only if  $\pi^* \omega$  has comass one pointwise along  $M$  where  $\omega$  is the induced volume form of  $M$ .*

**Proof.** For  $x \in M$ , take an oriented orthonormal basis  $\{e_1, \dots, e_m\}$  of  $T_x M$ . Then we have unique decompositions  $e_i = \sin \theta_i \cdot a_i + \cos \theta_i \cdot b_i$  where  $b_i$  is some unit vector in  $F_x$  – the subspace of fiber directions in  $T_x E$ ,  $a_i$  is a unit vector perpendicular to  $F_x$ , and  $\theta_i$  is the angle between  $e_i$  and  $F_x$ . By the choice of  $\{e_i\}$ ,

$$(3.3) \quad \begin{aligned} 1 &= \omega(e_1 \wedge e_2 \cdots \wedge e_m) \\ &= \pi^* \omega(e_1 \wedge e_2 \cdots \wedge e_m) \\ &= \pi^* \omega(\sin \theta_1 \cdot a_1 \wedge \cdots \wedge \sin \theta_m \cdot a_m) \\ &= \prod \sin \theta_i \cdot \pi^* \omega(a_1 \wedge a_2 \cdots \wedge a_m). \end{aligned}$$

The third equality is because that elements of  $F_x$  annihilate  $\pi^*\omega$ . Since  $\{a_i\}$  are of unit length,  $\|\pi^*\omega\|_{x,g}^* \geq 1$ ,  $\forall x \in M$ . By Remark 2.2, the equality holds if and only if  $F_x \perp T_x M$ .  $\blacksquare$

**Remark 3.5.** Since  $\pi^*\omega$  is smooth and simple,  $\|\pi^*\omega\|_g^*$  is smooth. By Lemma 3.1,  $(\pi^*\omega, (\|\pi^*\omega\|_g^*)^{\frac{2}{m}}g)$  is a calibration pair of  $M$  on  $E$ .

**3.2. Global forms.** In the singular homology theory the *Kronecker* product  $\langle \cdot, \cdot \rangle$  between cochains and chains induces a homomorphism

$$\kappa : H^q(X; G) \rightarrow \text{Hom}_{\mathbb{Z}}(H_q(X; \mathbb{Z}), G) \text{ given by}$$

$$\kappa([z^q])([z_q]) \triangleq \langle [z^q], [z_q] \rangle$$

where  $G$  is an Abelian group. A classical result asserts that  $\kappa$  is surjective. When  $G = \mathbb{R}$ , by the de Rham Theorem,  $\kappa : H_{dR}^q(X) \rightarrow \text{Hom}_{\mathbb{R}}(H_q(X; \mathbb{R}), \mathbb{R})$ .

Suppose  $\{M_\alpha\}$  are mutually disjoint  $m$ -dimensional oriented connected compact submanifolds with homology classes  $\{[M_\alpha]\}$  lying in one common side of some hyperplane through the zero of  $H_m(X; \mathbb{R})$ . Then there exists a homomorphism  $F \in \text{Hom}_{\mathbb{R}}(H_m(X; \mathbb{R}), \mathbb{R})$  forwarding  $\{[M_\alpha]\}$  to positive numbers. As a consequence, we have the following.

**Lemma 3.6.** Suppose  $\{M_\alpha\}$  satisfy the above condition. Then there exists a closed  $m$ -form  $\phi$  on  $X$  with  $\int_{M_\alpha} \phi > 0$  for each  $M_\alpha$ .

**3.3. Gluing of forms.** Given an oriented connected compact submanifold  $M$  in  $(X, g)$ , consider its  $\epsilon$ -neighborhood  $U_\epsilon$ . When  $\epsilon$  is small enough, the metric induces a disk bundle structure of  $U_\epsilon$ , whose fiber is given by the exponential map restricted to normal directions of  $M$ . Hence by Remark 3.5 a local calibration pair of  $M$  can be produced. We shall extend (a modification of) this local pair to a global one. Let us glue forms first.

By a strong deformation retraction from  $U_\epsilon$  to  $M$ ,  $H^m(U_\epsilon; \mathbb{R}) \cong H^m(M; \mathbb{R})$ . Therefore for any  $[\phi_1]$  and  $[\phi_2] \in H^m(U_\epsilon; \mathbb{R})$

$$(3.4) \quad [\phi_1] = [\phi_2] \Leftrightarrow \int_M \phi_1 = \int_M \phi_2.$$

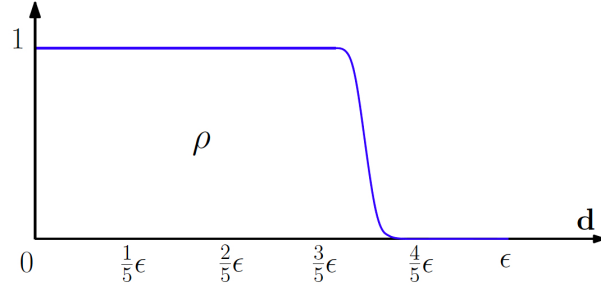
Assume further  $[M] \neq [0] \in H_m(X; \mathbb{R})$ . By §3.2 there exists a closed  $m$ -form  $\phi$  on  $X$  with  $s = \int_M \phi > 0$ . Let  $\pi$  be the projection map of the disk bundle. Then in  $U_\epsilon$

$$(3.5) \quad \int_M \frac{s \cdot \pi^* \omega}{\text{Vol}_g(M)} = s = \int_M \phi$$

where  $\omega$  is the volume form of  $M$ . Denote the integrand of the left hand side of (3.5) by  $\omega^*$ . By (3.4)  $[\omega^*] = [\phi]$  in  $H^m(U_\epsilon; \mathbb{R})$  which indicates

$$(3.6) \quad \phi = \omega^* + d\psi$$

for some smooth  $(m-1)$ -form  $\psi$  on  $U_\epsilon$ . Now take  $\Phi = \omega^* + d((1 - \rho(\mathbf{d}))\psi)$  where  $\mathbf{d}$  is the distance function to  $M$  and  $\rho$  is given in the picture. Clearly  $\Phi$  extends to a closed smooth form



on  $X$ :

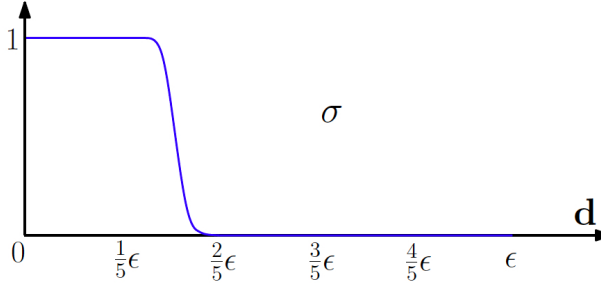
$$\Phi = \begin{cases} \omega^* & 0 \leq \mathbf{d} \leq \frac{3}{5}\epsilon \\ \omega^* + d((1 - \rho(\mathbf{d}))\psi) & \frac{3}{5}\epsilon < \mathbf{d} \leq \frac{4}{5}\epsilon \\ \phi & \frac{4}{5}\epsilon < \mathbf{d} \end{cases}$$

By Remark 3.5  $(\Phi, (\|\Phi\|_g^*)^{\frac{2}{m}}g)$  is a calibration pair of  $M$  for  $\mathbf{d} < \frac{3}{5}\epsilon$ .

**3.4. Gluing of metrics.** Our goal is to extend  $(\|\Phi\|_g^*)^{\frac{2}{m}}g$  to a global metric under which the global form  $\Phi$  becomes a calibration. Choose an appropriate positive smooth function  $\alpha$  such that

$$(3.7) \quad \|\Phi\|_{\alpha g}^* < 1 \text{ on } X,$$

and a gluing function  $\sigma = \sigma(\mathbf{d})$  shown in the picture. Then by Lemma 3.3



$$(3.8) \quad \tilde{g} = \sigma^{\frac{1}{m}}(1 + \mathbf{d}^2)(\|\Phi\|_g^*)^{\frac{2}{m}}g + \alpha(1 - \sigma)^{\frac{1}{m}}g$$

can serve for our purpose. Here the factor  $(1 + \mathbf{d}^2)$  makes  $\mathbf{spt}(\|\Phi\|_{\tilde{g}}^* - 1) = M$  which implies that  $[[M]]$  is uniquely mass-minimizing in  $[M]$ .

**3.5. Some results.** We can have a few consequences of the constructions in §3.3 and §3.4. An immediate one is this.

**Theorem 3.7.** *Suppose  $(X, g)$  is a Riemannian manifold and  $M$  is an oriented connected compact  $m$ -dimensional submanifold with  $[M] \neq 0 \in H_m(X; \mathbb{R})$ . Then there exists a metric  $\hat{g}$  conformal to  $g$  such that  $[[M]]$  is the unique mass-minimizer in its homology class in  $(X, \hat{g})$ .*

**Remark 3.8.** When  $X$  is compact,  $\alpha$  in (3.7) can be taken as a sufficiently large constant. Set  $\hat{g} \triangleq \alpha^{-1}\tilde{g}$  and  $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}}\Phi$ . Then  $(\hat{\Phi}, \hat{g})$  is a calibration pair of  $M$  and  $\hat{g} = g$  on  $X - U_\epsilon$ .

Assume that  $M$  is an oriented submanifold with (countably many) connected components  $\{M_i\}$  and that every  $M_i$  is compact. If  $\{M_i\}$  satisfies the condition in §3.2, then the same procedure works and we have the following.

**Theorem 3.9.** Let  $M$  be given as above in  $(X, g)$ . Then there exist a metric  $\hat{g}$  conformal to  $g$  and a calibration  $\hat{\Phi}$  such that every nonzero current  $T = \sum_i t_i [[M_i]]$ , where  $\{t_i\}$  are nonnegative and only finitely many of them are nonzero, is calibrated in  $(X, \hat{\Phi}, \hat{g})$ .

When each  $[M_i]$  is nonzero, one can choose some hyperplane  $\mathcal{P}_m$  through zero in  $H_m(X; \mathbb{R})$  that avoids all classes  $\{[M_i]\}$ . Now  $\mathcal{P}_m$  divides the space into two open chambers. By reversing orientations of components in one chamber, we get a new collection satisfying the requirement in §3.2.

**Corollary 3.10.** Suppose each  $[M_i]$  is nonzero. Then in any conformal class of metrics there exists a metric  $\hat{g}$  such that every  $[[M_i]]$  is homologically mass-minimizing in  $(X, \hat{g})$ .

In order to have a clearer description in more general situation, we need some definitions.

**Definition 3.11.** A family  $\mathfrak{M}$  of mutually disjoint oriented connected compact submanifolds of  $X$  is called a **mutually disjoint collection** and an element of  $\mathfrak{M}$  is a **component**. The (nonempty) subset  $\mathfrak{M}_k$  of all components of dimension  $k$  is its  **$k$ -level**.

**Definition 3.12.** Let  $\mathfrak{M} = \{M_i\}_{i=1,2,\dots}$  be a mutually disjoint collection of countably many components. If the set  $\bigcup_{i \neq j} M_i$  is closed for every  $j$ , then  $\mathfrak{M}$  is called a **neat collection**.

The neatness implies the existence of  $\epsilon_i > 0$  such that  $\{U_{\epsilon_i}(M_i)\}$  are mutually disjoint.

**Theorem 3.13.** Suppose that  $\mathfrak{M}$  is a neat collection and that each component represents a nonzero class in the  $\mathbb{R}$ -homology of  $X$ . In addition, assume every level of  $\mathfrak{M}$  has finite components except the lowest level. Then in any conformal class of metrics there exists a metric  $\hat{g}$  such that each  $[[M_i]]$  is homologically mass-minimizing in  $(X, \hat{g})$ .

**Proof.** Without loss of generality, let  $\mathfrak{M} = \{A^a, B^b\}$  with  $a > b$  and  $g$  be a metric. Take small positive  $\epsilon_{1,2}$  for the procedure in §3.3 so that  $U_{\epsilon_1}(A)$  and  $U_{\epsilon_2}(B)$  are disjoint. Suppose one gets an  $a$ -form  $\Phi$  for  $A$ . Then  $\Phi = d\theta$  in  $U_{\epsilon_2}(B)$  for some form  $\theta$  of degree  $a-1$ . So  $\Phi$  can be assumed identically zero in  $U_{\epsilon_2}(B)$  from the beginning. Using an  $\alpha_a$  whose value remains one on  $U_{\epsilon_2}(B)$  we get a metric  $\tilde{g}$  by §3.4 under which  $A$  is calibrated by  $\Phi$ .

By the compactness of  $\overline{U_{\epsilon_1}(A)}$ , there is a  $b$ -form  $\psi$  with  $\int_B \psi > 0$  and  $\|\psi\|_{\tilde{g}}^* < 1$  on  $\overline{U_{\epsilon_1}(A)}$ . Suppose we get  $\Psi$  following §3.3. Then one can use an  $\alpha_b \geq 1$  with value one in  $\overline{U_{\epsilon_1}(A)}$  for

$$\|\Psi\|_{\alpha_b \tilde{g}}^* < 1 \text{ on } X.$$

By §3.4 we get a calibration pair  $(\Psi, \hat{g})$  of  $B$ . Note that  $(\Phi, \hat{g})$  is a calibration of  $A$ . ■

**Remark 3.14.** The compactness of  $\overline{U_{\epsilon_1}(A)}$  is important. If a level of  $\mathfrak{M}$  has infinitely many components, then our current proof cannot descend further from that level.

In [Tas93] Tasaki studied the “equivariant” case.



**Theorem 3.15** (Tasaki). *Let  $K$  be a connected compact Lie transformation group of a manifold  $X$  and  $M$  be a (connected) compact oriented submanifold in  $X$ . Assume  $M$  is invariant under the action of  $K$  and it represents a nonzero  $\mathbb{R}$ -homology class of  $X$ . Then there exists a  $K$ -invariant Riemannian metric  $g$  on  $X$  such that  $M$  is mass-minimizing in homology class with respect to  $g$ .*

By our method, one can improve the result.

**Theorem 3.16.** *Let  $K$  be a compact Lie transformation group of a manifold  $X$  and  $M$  be a connected compact oriented submanifold with  $[M] \neq 0 \in H_m(X; \mathbb{R})$ . Assume  $M$  is invariant under the action of  $K$  and the action is orientation preserving. Then for any  $K$ -invariant Riemannian metric  $g^K$ , there exists a  $K$ -invariant metric  $\hat{g}^K$  conformal to  $g^K$  such that  $M$  can be calibrated in  $(X, \hat{g}^K)$ .*

**Proof.** There is a Haar-measure  $d\mu$  with  $\int_K d\mu = 1$  for compact  $K$ . Since the action is orientation preserving and  $g^K$  is  $K$ -invariant,  $\omega^*$  and  $\mathbf{d}$  are  $K$ -invariant. So one can use  $d\mu$  to average (3.6) for a  $K$ -invariant  $\Phi$  which equals  $\omega^*$  in  $M$ . Then average the corresponding  $\alpha$ . By (3.8) one can get a  $K$ -invariant calibration pair  $(\Phi, \hat{g}^K)$ . ■

Similarly one can have another generalization when  $K$  is connected.

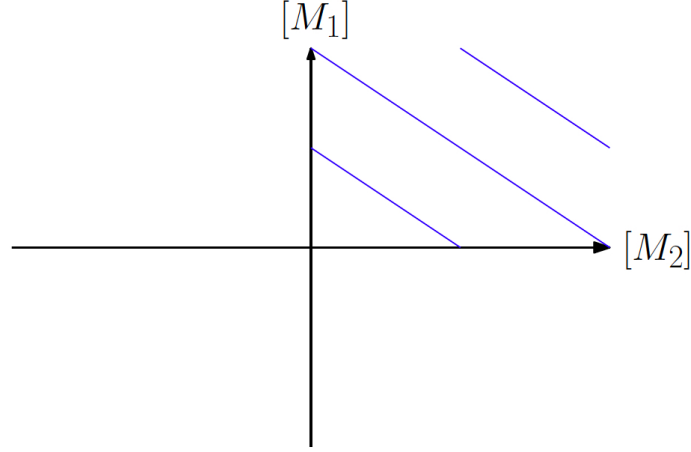
**Theorem 3.17.** *Suppose that  $\mathfrak{M}$  is a neat collection with only the lowest level possibly consisting of infinite components, and that each component represents a nonzero class in the  $\mathbb{R}$ -homology of  $X$ . Let  $K$  be a connected compact Lie transformation group of  $X$ . Assume  $\mathfrak{M}$  is invariant under the action of  $K$ . Then for any  $K$ -invariant Riemannian metric  $g^K$ , there exists a  $K$ -invariant metric  $\hat{g}^K$  conformal to  $g^K$  under which each component of  $\mathfrak{M}$  is homologically mass-minimizing.*

**3.6. More results.** Since only one calibration is constructed for each dimension, results in §3.5, e.g. Theorem 3.9, lack the control on some region of the space of homology classes. To conquer this, we shall construct a metric that supports enough calibrations we need.

When  $X^n$  is oriented with betti number  $b_k < \infty$  for  $1 \leq k < \frac{1}{2}n$ , by Thom [Tho54] or Corollary II.30 in [Tho07] there exist embedded oriented connected compact  $k$ -dimensional submanifolds  $\mathcal{L}_k \triangleq \{M_1^k, \dots, M_{b_k}^k\}$  such that  $\text{span}\{[M_i^k]\}_{i=1}^{b_k} = H_k(X; \mathbb{R})$ . By dimension reason one can arrange  $\bigcup_{1 \leq k < \frac{1}{2}n} \mathcal{L}_k$  to be a mutually disjoint collection.

**Theorem 3.18.** *Let  $M_i^k$  be given as above. Then in any conformal class of metrics there exists  $\hat{g}$  under which every nonzero  $\sum_{i=1}^{b_k} t_i [M_i^k]$  where  $1 \leq k < \frac{1}{2}n$ ,  $M_i^k \in \mathcal{L}_k$  and  $t_i \in \mathbb{R}$  is the unique mass-minimizing current in  $\sum_{i=1}^{b_k} t_i [M_i^k]$ .*

**Proof.** For the sake of simplicity, assume  $\dim H_k(X; \mathbb{R}) = 2$  for some  $k < \frac{1}{2}n$  and  $\{[M_1], [M_2]\}$  is a basis where  $M_1$  and  $M_2$  are disjoint oriented connected compact submanifolds. Then there exist  $k$ -forms  $\phi_1$  and  $\phi_2$  on  $X$  with  $\int_{M_i} \phi_j = \delta_{i,j}$ . Without loss of generality, assume  $\phi_1 \equiv 0$  on  $U_\epsilon(M_2)$  and  $\phi_2 \equiv 0$  on  $U_\epsilon(M_1)$  for some small  $\epsilon$ . Note that  $\alpha$  can be chosen so that  $\|\hat{\Phi}_i\|_{\hat{g}}^* \leq \frac{1}{2}$  on  $(U_\epsilon(M_i))^c$  for the resulting forms  $\hat{\Phi}_1$  and  $\hat{\Phi}_2$  in §3.3 under the metric  $\hat{g}$  in §3.4. A key observation is that  $\pm \hat{\Phi}_1$ ,  $\pm \hat{\Phi}_2$  and  $\pm \hat{\Phi}_1 \pm \hat{\Phi}_2$  are all calibrations with respect to  $\hat{g}$ .



Then any nonzero linear combination of  $[[M_1]]$  and  $[[M_2]]$  can be calibrated in  $(X, \hat{g})$ . For example, those representing classes of the closer of the first quadrant can be calibrated by  $\hat{\Phi}_1 + \hat{\Phi}_2$ . The uniqueness follows as a result of  $\text{spt}(\|\pm \hat{\Phi}_i\|_g^* - 1) = M_i$ ,  $\text{spt}(\|\pm \hat{\Phi}_1 \pm \hat{\Phi}_2\|_g^* - 1) = \bigcup M_i$ , the simpleness of  $\pm \hat{\Phi}_i$  along  $M_i$  and  $\pm \hat{\Phi}_1 \pm \hat{\Phi}_2$  along  $M_1 \cup M_2$ , and the connectedness of  $M_i$ .

When  $\dim H_k(X; \mathbb{R}) = s$ ,  $2^s$  such calibrations, each of which has comass norm bounded above by  $\frac{1}{s}$  away from some neighborhood of corresponding submanifold, can be constructed for our purpose. More generally, for different dimension levels, the above argument combined with the proof of Theorem 3.13 proves the theorem.  $\blacksquare$

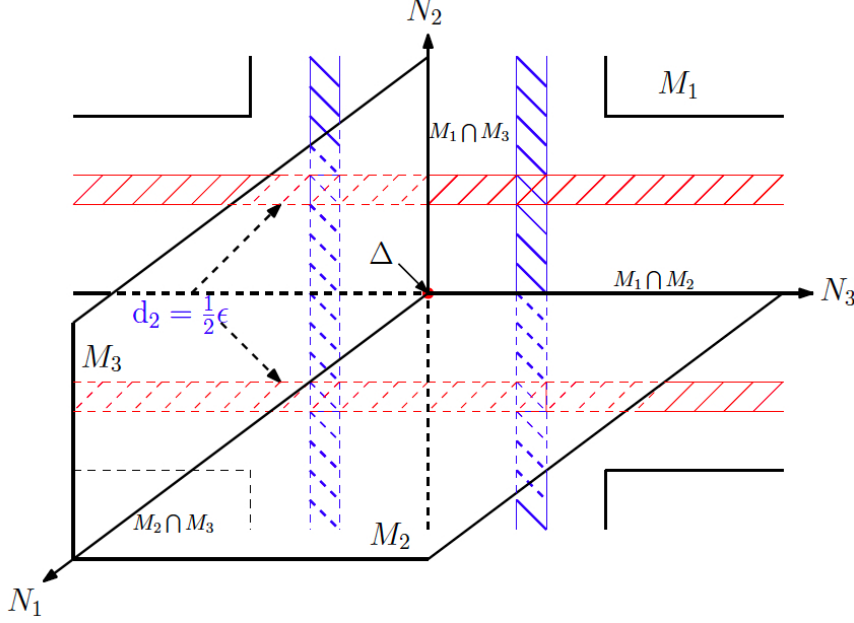
When  $\dim(X) \geq 6$ , one can choose  $b_k$  smooth  $k$ -dimensional submanifolds  $\mathcal{L}_k = \{M_i^k\}_{i=1}^{b_k}$  for  $k = 1, 2, \dots, n-3$ , such that  $\text{span}\{[M_i^k]\}_{i=1}^{b_k} = H_k(X, \mathbb{R})$  and such that intersections  $\mathcal{I}$  among  $\bigcup_{k=1}^{n-3} \mathcal{L}_k$  are all transversal. Note that  $\mathcal{I}$  has a natural stratification structure  $\dots < \mathcal{I}_2 < \mathcal{I}_1 = \bigcup_{k=1}^{n-3} \mathcal{L}_k$ , where  $\mathcal{I}_t$  is the set of intersections among  $t$  representatives.

**Theorem 3.19.** *Let  $X^n$  be an oriented manifold with betti numbers  $b_k < \infty$  for  $1 \leq k \leq n-3$  and  $\mathcal{L}_k$  given above. Then there exists a metric  $g$  such that every nonzero  $\sum_{i=1}^{b_k} t_i [[M_i^k]]$  where  $1 \leq k \leq n-3$ ,  $M_i^k \in \mathcal{L}_k$  and  $t_i \in \mathbb{R}$  is the unique mass-minimizing current in  $\sum_{i=1}^{b_k} t_i [M_i^k]$ .*

**Proof.** One can build a metric  $g$  on  $X$  such that, for any element  $S$  of  $\mathcal{I}_t$  ( $t \geq 2$ ), there exists some  $2\epsilon$ -cubic neighborhood of  $S$  with fibers (induced by  $g$  as in §3.3) split pointwise along  $S$  as the Riemannian product of fibers of  $2\epsilon$ -cubic neighborhoods of  $S$  in  $H_S$  for all  $H_S \in \mathcal{I}_{t-1}$  and  $S \subseteq H_S$ .

Let us focus on all (connected parts of) deepest intersections. For simplicity, suppose we have only one connected deepest intersection  $\Delta$  and  $\Delta \in \mathcal{I}_3$ . Namely  $\Delta$  is the intersection of three submanifolds  $M_1$ ,  $M_2$  and  $M_3$ . Assume  $2\epsilon$  is universal for  $S \in \bigcup_{t \geq 2} \mathcal{I}_t$  under  $g$  in the preceding paragraph. Denote the volume form of  $M_3$  by  $\omega_3$ , the distance function to  $M_3$  by  $\mathbf{d}_3$ , and the projection to nearest point on  $M_3$  by  $\pi_3$ .  $\omega_1, \mathbf{d}_1, \pi_1$  and  $\omega_2, \mathbf{d}_2, \pi_2$  are similarly defined. Since  $\omega_i = d\psi_i$  in the  $\epsilon$ -neighborhood of  $(M_i \cap M_{i+1}) \cup (M_i \cap M_{i+2})$  in  $M_i$  (subscripts in the sense of mod 3), define  $\Psi_i = d(\rho_i \psi_i)$  in the union of  $\epsilon$ -cubic neighborhoods of  $M_i \cap M_{i+1}$  and  $M_i \cap M_{i+2}$ . Here we identify the pullback of  $\omega_i$  (and  $\psi_i$ ) via  $\pi_i$  with itself, and  $\rho_i$  is a smooth

increasing function in  $\mathbf{d}_i$  with value zero when  $\mathbf{d}_i \leq \frac{1}{2}\epsilon$  and value one for  $\frac{2}{3}\epsilon \leq \mathbf{d}_i \leq \epsilon$ . The slash-shadow region and the backslash-shadow region are intersections of regions  $\Gamma_2 : \frac{1}{2}\epsilon \leq \mathbf{d}_2 \leq \frac{2}{3}\epsilon$  and  $\Gamma_3 : \frac{1}{2}\epsilon \leq \mathbf{d}_3 \leq \frac{2}{3}\epsilon$  with  $M_1$  respectively.



**Case One:** Every  $M_i$  has the same dimension  $k$ . There are three (bunches of) directions  $N_i$  on the  $\epsilon$ -cubic neighborhood  $U_\epsilon(\Delta)$  of  $\Delta$ . (Note that  $N_i$  has meanings for  $\mathbf{d}_i \leq 2\epsilon$  only.) Denote the split part of  $g$  along  $N_i$  by  $g_i$ .

**Claim:**  $\sum \omega_i$  is a calibration in  $U_\epsilon(\Delta)$ .

Since a form and its Hodge dual have the same comass, the claim is an immediate consequence by applying the following lemma to  $\sum \omega_i = \sum * \omega_i$ .

**Lemma 3.20.** *Let  $e_1, \dots, e_{n+2}$  be an orthonormal basis for  $\mathbb{R}^{n+2}$ , and for each multi-index  $I = \{i_1, \dots, i_p\}$  where  $i_1 < \dots < i_p$ , let  $e_I^*$  denote the corresponding “axis”  $p$ -form  $e_{i_1}^* \wedge \dots \wedge e_{i_p}^*$ . Assume  $\phi = e_J^* \wedge e_{n+1}^* \wedge e_{n+2}^*$  where  $J = \{j_1, \dots, j_{p-2}\} \subset \{1, \dots, n\}$  and  $\psi = \sum_I e_I^*$  with  $i_p \leq n$ . Then*

$$\|\phi + \psi\|^* = \max\{1, \|\psi\|^*\}.$$

However  $\sum \omega_i$  is not well defined on the union  $\Xi$  of  $\epsilon$ -cubic neighborhoods of  $M_1 \cap M_2$ ,  $M_2 \cap M_3$  and  $M_3 \cap M_1$ . Instead we consider

$$\phi_k = \sum \omega_i - \sum \Psi_i = \sum [(1 - \rho_i)\omega_i - d\rho_i \wedge \psi_i] \text{ on } \Xi.$$

Then

$$(3.9) \quad \phi_k = \omega_i \text{ when } \mathbf{d}_i \leq \frac{1}{2}\epsilon, \mathbf{d}_{i+1} \geq \frac{2}{3}\epsilon \text{ and } \mathbf{d}_{i+2} \geq \frac{2}{3}\epsilon.$$

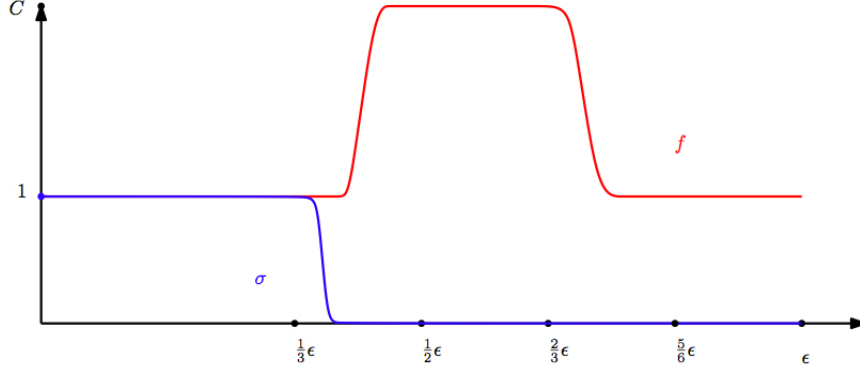
Note that, under the condition  $n - k \geq 3$ , for example in  $M_1$ , the subspace spanned by the dual of  $(1 - \rho_i)\omega_i - d\rho_i \wedge \psi_i$  for  $i \neq 1$  contains at least 2 directions of  $N_1$ . So, by the useful lemma of Harvey and Lawson below, if one multiples  $g_1$  by a sufficiently large constant  $C > 1$ , then  $\phi_k$  has comass one (same as that of  $\omega_1$ ) in  $\Xi \cap M_1$ .

**Lemma 3.21** (Corollary 2.11. in [HL82b]). *With notation as in Lemma 3.20,*

$$\|e_1^* \wedge \cdots \wedge e_p^* + \sum_I b_I e_I^*\|^* \leq \max\{1, \sum_I |b_I|\}$$

provided that  $b_I = 0$  whenever  $i_{p-1} \leq p$ .

We are now about to modify  $g$  so that  $\phi_k$  becomes a calibration in some neighborhood of  $\bigcup M_i$ . Let  $C$  work for each  $\Xi \cap M_i$ . Choose a smooth function  $f$  of  $\mathbf{d}$  for  $\mathbf{d} \leq \epsilon$  as in the picture and set  $f_i = f(\mathbf{d}_i)$ .



Along  $M_1 \cap M_2$ , set

$$g_1 \rightarrow f_3 g_1, \quad g_2 \rightarrow f_3 g_2, \quad \text{and} \quad g_3 \rightarrow f_3^{-1} g_3 \quad (\star)$$

and similarly for  $M_2 \cap M_3$  and  $M_3 \cap M_1$ . Then in these three sets  $\phi_k$  becomes a calibration.

We want to extend the metric along each  $M_i$ . A good try based on  $(\star)$  to the  $\epsilon$ -neighborhood of  $M_1 \cap M_2$  in  $M_1$  is this.

$$g_1 \rightarrow f_3 g_1, \quad g_2 \rightarrow f_3^{\sigma_2} g_2, \quad \text{and} \quad g_3 \rightarrow f_3^{-\sigma_2} g_3 \quad (*)$$

where  $\sigma$  is a cutoff function with  $f = 1$  on  $\mathbf{spt}(\sigma)$  and  $\sigma_i = \sigma(\mathbf{d}_i)$ . A subtle point here is that the volume form of  $M_1$  is unchanged. The same extension from  $M_1 \cap M_3$  to  $M_1$  gives the following.

$$g_1 \rightarrow f_2 g_1, \quad g_2 \rightarrow f_2^{-\sigma_3} g_2, \quad \text{and} \quad g_3 \rightarrow f_2^{\sigma_3} g_3.$$

Since these two extension do not agree in  $U_\epsilon(\Delta) \cap M_1$ , we combine them by

$$g_1 \rightarrow f_2 f_3 g_1, \quad g_2 \rightarrow f_2^{-\sigma_3} f_3^{\sigma_2} g_2, \quad \text{and} \quad g_3 \rightarrow f_2^{\sigma_3} f_3^{-\sigma_2} g_3.$$

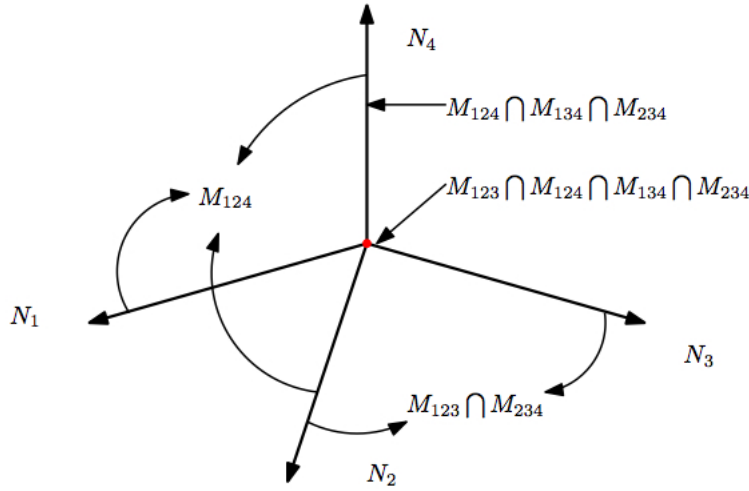
By  $f \geq 1$  and  $f = 1$  on  $\mathbf{spt}(\sigma)$ , the above combination will not affect the comass of  $\phi_k$  being one. Together with the same procedure for  $M_2$  and  $M_3$ , we get a metric  $\tilde{g}$  in  $\bigcup \Xi \cap M_i$  that makes  $\phi_k$  a calibration. Note that in the  $\frac{1}{3}\epsilon$  neighborhoods of  $M_{i+1}$  and  $M_{i+2}$  in  $M_i$ , or in the complement of the intersection of  $\frac{5}{6}\epsilon$  neighborhoods of  $M_{i+1}$  and  $M_{i+2}$  in  $U_\epsilon(\Delta) \cap M_i$ ,  $\tilde{g} = g$ . Hence by (3.9)  $\tilde{g}$  produces a metric  $\check{g}$  on the union  $\Upsilon$  of  $\frac{1}{3}\epsilon$  neighborhoods of  $M_i$  (containing each  $M_i$ ) making  $\phi_k$  a calibration. Furthermore every nonzero  $\sum_{i, n_i = \pm 1, 0} n_i (\omega_i - \Psi_i)$  becomes a calibration in  $(\Upsilon, \check{g})$ .

**Case Two:**  $M_2$  and  $M_3$  are of dimension  $k$ , but  $M_1$  has a different dimension  $m$ . (Similar for the case with mutually different dimensions.) Consider potential calibrations  $\pm(\omega_2 - \Psi_2)$ ,  $\pm(\omega_3 - \Psi_3)$ ,  $\pm(\omega_2 - \Psi_2) \pm (\omega_3 - \Psi_3)$ , and  $\pm(\omega_1 - \Psi_1)$  on  $\Xi$ . By the same procedure (but with different

weights in  $(\star)$  and  $(*)$ ), one can get calibration pairs on some neighborhood of  $\bigcup M_i$ .

The idea works for general cases with modified  $(\star)$  and  $(*)$ . Following the above steps around all connected parts of deepest intersection, one can extend the preferred local calibration pairs to global ones sharing a common metric. Multiply the metric by a smooth function which is one in  $\bigcup_{k=1}^{n-3} \bigcup_{M \in \mathcal{L}_k} M$  and strictly greater than one elsewhere. Name it  $\hat{g}$ . Then every nonzero  $\sum_{t_i \in \mathbb{R}, M_i \in \mathcal{L}_k} t_i [[M_i]]$  with  $1 \leq k \leq n-3$  can be calibrated in  $(X, \hat{g})$ . The uniqueness of such a mass-minimizing current in its current homology class follows similarly as in the proof of Theorem 3.19. Here note that for any point  $p \in M_i^k - \mathcal{I}_2$  (a.e. on  $M_i^k$ ) the oriented unit  $k$ -vector of  $\wedge^k T_p M_i^k$  is the unique unit  $k$ -vectors in  $\wedge^k T_p X$  that has pairing value one with the corresponding calibrations of  $M_i^k$ .  $\blacksquare$

**Remark 3.22.** To show how  $(\star)$  and  $(*)$  change, suppose in Case One we have  $M_{123}$ ,  $M_{124}$ ,  $M_{134}$ ,  $M_{234}$  and perpendicular directions  $N_4$ ,  $N_3$ ,  $N_2$ ,  $N_1$  respectively given in the figure below.



Then, along  $M_{124} \cap M_{134} \cap M_{234}$ ,  $(\star)$  transforms to

$$g_1 \rightarrow f_4 g_1, \quad g_2 \rightarrow f_4 g_2, \quad g_3 \rightarrow f_4 g_3 \quad \text{and} \quad g_4 \rightarrow f_4^{-2} g_4 \quad (\star')$$

and a good try of metric extension to  $M_{124}$  is

$$g_1 \rightarrow f_4^{\sigma_1 \sigma_2} g_1, \quad g_2 \rightarrow f_4^{\sigma_1 \sigma_2} g_2, \quad g_3 \rightarrow f_4 g_3 \quad \text{and} \quad g_4 \rightarrow f_4^{-2\sigma_1 \sigma_2} g_4 \quad (*').$$

The corresponding  $\tilde{g}$  in  $U_\epsilon(\Delta) \cap M_{124}$  is given by

$$\begin{aligned} g_1 &\rightarrow f_1^{-2\sigma_2 \sigma_4} f_2^{\sigma_1 \sigma_4} f_4^{\sigma_1 \sigma_2} g_1, \\ g_2 &\rightarrow f_1^{\sigma_2 \sigma_4} f_2^{-2\sigma_1 \sigma_4} f_4^{\sigma_1 \sigma_2} g_2, \\ g_3 &\rightarrow f_1 f_2 f_4 g_3, \quad \text{and} \\ g_4 &\rightarrow f_1^{\sigma_2 \sigma_4} f_2^{\sigma_1 \sigma_4} f_4^{-2\sigma_1 \sigma_2} g_4. \end{aligned}$$

**Remark 3.23.** In general codimension at least 3 is vital to apply Lemma 3.21. For  $n = 4$  or  $5$ , Theorem 3.19 can be improved to include the level of codimension 2 by Theorem 4.6.

**Proof of Lemma 3.20.** Assume the comass of  $\phi + \psi$  is achieved by pairing with a unit  $p$ -vectors  $\xi$ . Then we will make use of the following “canonical form of a simple vector with respect to a subspace”.

**Lemma 3.24** (Lemma 7.5 in [HL82a]). *Suppose  $V \subset \mathbb{R}^n$  is a linear subspace and  $\xi$  is a unit simple  $p$ -vector. Then there exists set of orthonormal vectors  $f_1, \dots, f_r$  in  $V$ , a set of orthonormal vectors  $g_1, \dots, g_s$  in  $V^\perp$ , and angles  $0 < \theta_j < \frac{\pi}{2}$  for  $j = 1, \dots, k$  (where  $k \leq r, s \leq p$  and  $r + s - k = p$ ) such that*

$$\xi = (\cos \theta_1 f_1 + \sin \theta_1 g_1) \wedge \dots \wedge (\cos \theta_k f_k + \sin \theta_k g_k) \wedge f_{k+1} \wedge \dots \wedge f_r \wedge g_{k+1} \wedge \dots \wedge g_s.$$

Let  $V = \text{span}\{e^{n+1}, e^{n+2}\}$ . These  $\lambda_j = \cos^2 \theta_j$  are eigenvalues of a symmetric bilinear form  $B$  where  $\pi : \mathbb{R}^m \rightarrow V$  and  $B(u, v) = \langle \pi(u), \pi(v) \rangle$  is defined on  $\text{span } \xi$ .

Assume  $r = k = 2$  and  $s = p$  (otherwise either  $\langle \phi, \xi \rangle$  or  $\langle \psi, \xi \rangle$  gives zero and a proof or contradiction follows easily). We have

$$\xi = (\cos \theta_1 e_1 + \sin \theta_1 g_1) \wedge (\cos \theta_2 e_2 + \sin \theta_2 g_2) \wedge g_3 \wedge \dots \wedge g_p.$$

Evaluating  $\phi + \psi$  on  $\xi$  shows

$$\begin{aligned} |\langle \phi + \psi, \xi \rangle| &= |\cos \theta_1 \cos \theta_2 \cdot \phi(e_1, e_2, g_3, \dots, g_p) + \sin \theta_1 \sin \theta_2 \cdot \psi(g_1, g_2, g_3, \dots, g_p)| \\ &\leq \cos \theta_1 \cos \theta_2 \cdot |\phi(e_1, e_2, g_3, \dots, g_p)| + \sin \theta_1 \sin \theta_2 \cdot |\psi(g_1, g_2, g_3, \dots, g_p)| \\ &\leq \cos(\theta_1 - \theta_2) \cdot \max\{1, \|\psi\|^*\} \\ &\leq \max\{1, \|\psi\|^*\} \end{aligned}$$

**Question:** Usually one cannot have such existence result when  $k$  can be  $n - 1$ . Therefore it may be interesting to ask whether the same conclusion holds for  $1 \leq k \leq n - 2$  in general. ■

#### 4. SINGULAR CASE

In this section the case of submanifolds with singularities will be discussed. Unlike the smooth case, one cannot have local calibration pairs so easily as in §3.1. Our concern here is to extend an existing local calibration pair around the singular set to a calibration pair on some neighborhood of the singular submanifold under consideration. Then a further extension from the neighborhood to global is roughly the same as in the smooth case.

We first recall two useful lemmas in [HL82b], then obtain our extension theorem, and finally apply it for several interesting examples in the realm of calibrated geometry.

**4.1. Two lemmas.** The first lemma tells us how to canonically decompose a  $p$ -form with respect to certain  $p$ -plane.

**Lemma 4.1** (Harvey and Lawson). *Let  $\xi \in \Lambda^p \mathbb{R}^n$  be a simple  $p$ -vector with  $V = \text{span}\{\xi\}$ . Suppose  $\phi \in \Lambda^p \mathbb{R}^n$  satisfies  $\phi(\xi) = 1$ . Then there exists a unique oriented complementary subspace  $W$  to  $V$  with the following property. For any basis  $v_1, \dots, v_n$  of  $\mathbb{R}^n$  such that  $\xi = v_1 \wedge \dots \wedge v_p$  and  $v_{p+1}, \dots, v_n$  is basis for  $W$ , one has that*

$$(4.1) \quad \phi = v_1^* \wedge \dots \wedge v_p^* + \sum a_I v_I^*,$$

where  $a_I = 0$  whenever  $i_{p-1} \leq p$ . Here  $I = \{i_1, \dots, i_p\}$  with  $i_1 < \dots < i_p$ .

The second lemma says how to create metrics based on the above decomposition with control on the comass of the form.

**Lemma 4.2** (Harvey and Lawson). *Let  $\phi$ ,  $V = \text{span}\{\xi\}$ , and  $W$  be given in Lemma 4.1. Consider an inner product  $\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^n$  such that  $V \perp W$  and  $\|\xi\| = 1$ . Choose any constant  $C^2 > \binom{n}{p} \|\phi\|^*$  and define a new inner product on  $\mathbb{R}^n = V \oplus W$  by setting  $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle_V + C^2 \langle \cdot, \cdot \rangle_W$ . Then under this new metric we have*

$$\|\phi\|^* = 1 \text{ and } \phi(\xi) = \|\xi\| = 1.$$

**Remark 4.3.** *If  $\phi(\xi) = \vartheta$  (positive) not necessarily one, one can apply Lemma 4.1 to  $\vartheta^{-1}\phi$  for  $\|\phi\|^* = \vartheta$ ,  $\|\xi\| = 1$  and  $\phi(\xi) = \vartheta$  by choosing  $C^2 > \vartheta^{-1} \binom{n}{p} \|\phi\|^*$ .*

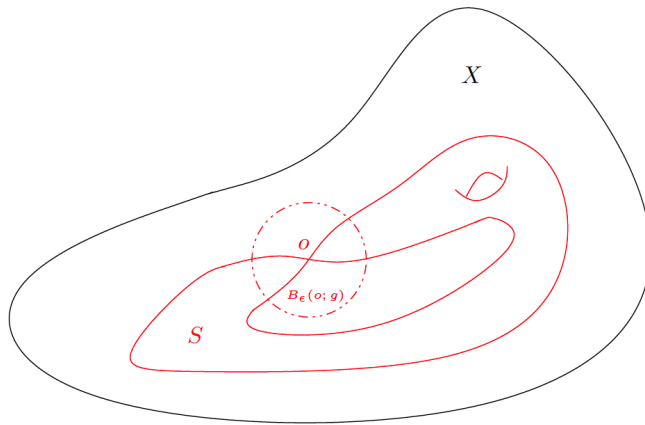
They will be used in proving the extension result in the next subsection.

#### 4.2. An extension result.

**Definition 4.4.** *By a **singular submanifold**  $(S, \mathcal{S})$  of dimension  $m$  with singular set  $\mathcal{S}$ , we mean a pair of closed subsets  $\mathcal{S} \subset S$  of  $X$ , where  $S - \mathcal{S}$  is an  $m$ -dimensional submanifold and the Hausdorff  $m$ -measure  $\mathcal{H}^m(\mathcal{S}) = 0$ .*

**Remark 4.5.** *Assume  $S$  is a submanifold with only one singular point  $p$  and  $C_p$  is a tangent cone of  $S$  at  $p$ . Then the current  $[[S]] = \int_S \cdot$  is calibrated by a smooth  $\phi$  if and only if  $S - p$  is calibrated by  $\phi$ . Moreover, either of them implies that  $\phi_p$  calibrates  $C_p$  in  $(T_p X, g_p)$ .*

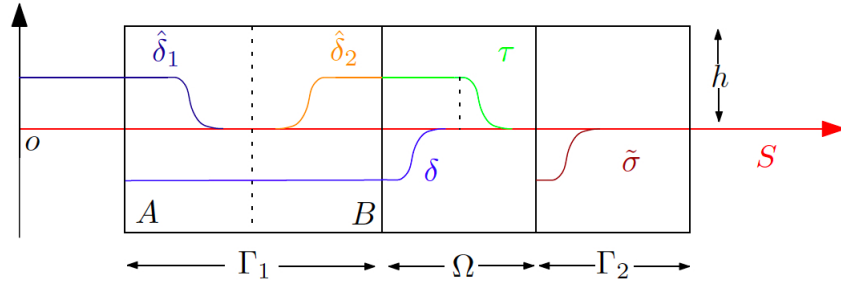
From now on,  $(S, o)$  will be assumed an oriented connected compact singular submanifold with one singular point  $o$ .



**Theorem 4.6.** Suppose  $(S, o) \subset (X, g)$  and  $[S] \neq [0] \in H_m(X; \mathbb{R})$ . If  $B_\epsilon(o; g) \cap S$  can be calibrated by a smooth calibration in some  $\epsilon$ -ball  $(B_\epsilon(o; g), g)$  centered at  $o$ , then there exists a metric  $\hat{g}$  coinciding with  $g$  on  $B_{\frac{\epsilon}{2}}(o; g)$  such that  $S$  can be calibrated by a smooth calibration in  $(X, \hat{g})$ .

**Remark 4.7.** In the theorem,  $\frac{\epsilon}{2}$  can be replaced by  $\kappa\epsilon$  for any  $0 < \kappa < 1$ .

**Proof.** Assume  $\epsilon$  is small enough so that the local calibration  $\phi$  on  $B_\epsilon(o; g)$  can be written as  $d\psi$  for some smooth  $(m-1)$ -form  $\psi$ . Suppose the compact region  $\Gamma_1 \cup \Omega \cup \Gamma_2$  (the diffeomorphic image of an  $h$ -disk normal bundle, for small  $h$ , over a closed set  $(\Gamma_1 \cup \Omega \cup \Gamma_2) \cap S$  by the exponential map restricted to normal directions, see picture below) is contained in  $B_\epsilon(o; g) - B_{\frac{2\epsilon}{3}}(o; g)$ . Denote the projection by  $\pi$  and call the directions perpendicular to fibers horizontal.



Then  $\pi^*\omega = d(\pi^*(\psi|_S))$  in  $\Gamma_1 \cup \Omega \cup \Gamma_2$  where  $\omega$  is the volume form of  $S \cap (\Gamma_1 \cup \Omega \cup \Gamma_2)$ . Set

$$\Phi \triangleq d(\tau\psi + (1-\tau)\pi^*(\psi|_S))$$

where  $\tau$  is a cut-off function on  $\Omega$  shown in the picture with value one near  $\Gamma_1$  and zero near  $\Gamma_2$ . (The picture is just an illustration, since “height”  $h$  is usually smaller than one.)

By shrinking  $h$ , the smooth function  $\Phi(\overrightarrow{T_y S_g}) > \frac{1}{2}$  on  $\Gamma_1 \cup \Omega \cup \Gamma_2$  where  $y \in \Gamma_1 \cup \Omega \cup \Gamma_2$  and  $\overrightarrow{T_y S_g}$  is the unique oriented unit horizontal  $m$ -vector at  $y$  with respect to  $g$ . Set

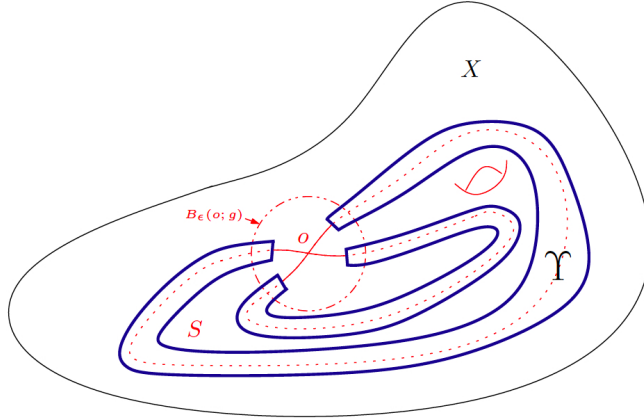
$$\bar{g} = f \cdot g \text{ where } f = \delta + (1-\delta)(\Phi(\overrightarrow{T_y S_g}))^{\frac{2}{m}}$$

on  $\Gamma_1 \cup \Omega \cup \Gamma_2$ . Note  $f = 1$  in  $S \cap (\Gamma_1 \cup \Omega \cup \Gamma_2)$  and  $\Phi = \phi$  on  $\text{spt}(\delta)$ . Since  $(\phi, g)$  is a local calibration pair,  $f \geq (\Phi(\overrightarrow{T_y S_g}))^{\frac{2}{m}}$  in  $\Gamma_1 \cup \Omega \cup \Gamma_2$  and  $f \equiv 1$  in  $\Gamma_1$ . Then  $\Phi$  and  $\bar{g}$  naturally extend on  $\Upsilon$ , the region embraced by the “curve” in the picture below (an “ $h$ -disk bundle” containing  $\Gamma_1 \cup \Omega \cup \Gamma_2$ ). Note that

- (a).  $\Phi$  calibrates  $S \cap (\Upsilon - \Omega)$  in  $(\Upsilon - \Omega, \bar{g})$ ,
- (b).  $\bar{g} = g$  in  $\Gamma_1$ , and
- (c).  $\frac{1}{2} < \Phi(\overrightarrow{T_y S_{\bar{g}}}) \leq 1$  on  $\Upsilon$  with equality on  $\Upsilon - \Gamma_1 - \Omega$ , where  $\overrightarrow{T_y S_{\bar{g}}}$  is the unique oriented unit horizontal  $m$ -vector at  $y$  for  $\bar{g}$ .

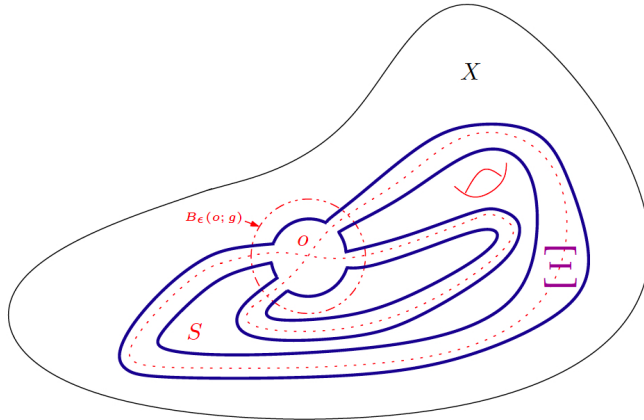
Now we wish to glue  $\bar{g}$  and  $g$  together to make  $\Phi$  a calibration. By applying Lemma 4.1 to  $\Phi$ ,  $\overrightarrow{T_y S_{\bar{g}}}$  and  $\bar{g}$  on  $\Upsilon$ , one can get a smoothly varying  $(n-m)$ -dimensional plane field  $\mathcal{W}$  transverse to the horizontal directions in  $\Upsilon$ . Following Lemma 4.2, Remark 4.3 and Property (c), for any





metric  $g_{\mathcal{W}}$  along  $\mathcal{W}$ , there exists a sufficiently large constant  $\bar{\alpha}$  (due to the compactness of  $\Upsilon$ ) such that, under  $\tilde{g} = \bar{g}^h \oplus \bar{\alpha}g_{\mathcal{W}}$  on  $\Upsilon$ , where  $\bar{g}^h$  is the horizontal part of  $\bar{g}$ ,

$$\|\Phi\|_{\tilde{g}}^* = \Phi(\overrightarrow{T_Y S_{\tilde{g}}}) \leq 1.$$



Based on Property (b) we construct a smooth metric  $\check{g}$  on  $\Xi$  as follows.

$$\check{g} = \begin{cases} g & \text{near } o \\ g + (1 - \hat{\delta}_1)((0 \cdot \bar{g}^h) \oplus \bar{\alpha}g_{\mathcal{W}}) & \text{on } A \\ (1 - \hat{\delta}_2)((0 \cdot g^h) \oplus g^\nu) + \tilde{g} & \text{on } B \\ \tilde{g} & \text{on } \Omega \\ \tilde{\sigma}\tilde{g} + (1 - \tilde{\sigma})\bar{g} & \text{on } \Gamma_2 \\ \bar{g} & \text{far away from } o \end{cases}$$

Here  $g^h, g^\nu$  are the horizontal and fiberwise parts of  $g$  respectively,  $\oplus$  means the orthogonal splitting of a (pseudo-)metric and  $+$  is the usual addition between two (pseudo-)metrics. Note that, on  $\Gamma_2$ ,  $\mathcal{W}$  is exactly the distribution of fiber directions and  $\Phi = \pi^*(\omega)$  is a simple horizontal  $m$ -form. So  $(\Phi, \check{g})$  becomes a calibration pair in  $\Xi$ . Since  $S$  is a strong deformation retract of  $\Xi$  and

$[S] \neq 0$ , it can extend to a global calibration pair of  $S$  by §3.3 and §3.4. ■

Since the comass function of a smooth form of co-degree one is always smooth, we have the following refinement.

**Corollary 4.8.** *Suppose  $(S, o)$  is of codimension one in  $(X, g)$  representing a nonzero real homology class. If  $B_\epsilon(o; g) \cap S$  for some  $\epsilon > 0$  can be calibrated by a coflat calibration singular only at  $o$  in  $(B_\epsilon(o; g), g)$ , then there exists a metric  $\hat{g}$  conformal to  $g$  with  $\hat{g} = g$  on  $B_{\frac{\epsilon}{2}}(o; g)$  such that  $S$  can be calibrated by a coflat calibration singular only at  $o$  in  $(X, \hat{g})$ .*

In fact it does not have to require that  $S$  is a strong deformation retract of some open neighborhood of  $S$  for the last step in the proof. Whenever there exists a global form that represents  $[\Phi]$  in some open neighborhood of  $S$ , our construction applies.

**Corollary 4.9.** *Suppose  $(S, \mathcal{S})$  is of dimension  $m$  in  $(X, g)$ . Assume  $V \cap S$  for some open neighborhood  $V$  of  $\mathcal{S}$  can be calibrated in  $(V, g|_V)$  by some coflat calibration  $\phi$  with singular set  $S_\phi \subset \mathcal{S}$ . Assume further  $[S] \neq [0]$  in  $H_m(X; \mathbb{R})$ . If*

$$i^* : H^m(X; \mathbb{R}) \rightarrow H^m(U; \mathbb{R}),$$

*is surjective for some neighborhood  $U$  of  $S$ , then there exists a metric  $\hat{g}$  such that  $S$  can be calibrated in  $(X, \hat{g})$  by a coflat calibration with singular set  $S_\phi$ .*

**Remark 4.10.** *By Almgren's big regularity theorem, being calibrated of  $S$  around  $\mathcal{S}$  implies that  $\mathcal{S}$  has codimension at least 2 in  $S$ . By  $\mathbf{spt}(d[[S]]) \subseteq \mathcal{S}$ ,  $d[[S]] = 0$  and therefore  $[S]$  makes sense.*

**Remark 4.11.** *When  $\mathcal{S}$  is a smooth submanifold,  $S$  is a strong deformation retract of  $B_\epsilon(S; g)$  for small  $\epsilon$ .*

**4.3. Further applications.** Under some circumstances calibrations cannot avoid having singularities. In [Zhaa] we showed that every homogeneous area-minimizing hypercones can have calibrations singular only at the origin.

**Example 1:** When the local model around  $o$  in Theorem 4.6 is a Simons cone over  $S^{r-1} \times S^{r-1}$  for  $r \geq 4$ , one has a smooth calibration  $\phi$  (which actually can be  $SO(r) \times SO(r)$  invariant) on  $\mathbb{R}^{2r} - \{0\}$ . Follow the proof of Theorem 4.6 to get  $\Phi$  on  $\Xi - o$  and  $\check{g}$  on  $\Xi$ . By Mayer-Vietoris sequence for  $\Xi - o$  and an open ball  $B$  with  $o \in B \subset \Xi$ , one gets the exact sequence

$$H^{2r-2}(S^{2r-1}) \rightarrow H^{2r-1}(\Xi) \rightarrow H^{2r-1}(\Xi - o) \rightarrow H^{2r-1}(S^{2r-1}) \rightarrow H^{2r}(\Xi),$$

where  $S^{2r-1}(v)$  is a small  $v$ -sphere centered at  $o$ . Since

$$\left\| \int_{S^{2r-1}(v)} \Phi \right\| = \left\| \int_{S^{2r-1}(v)} \phi \right\| = \lim_{v \rightarrow 0} \left\| \int_{S^{2r-1}(v)} \phi \right\| \leq \lim_{v \rightarrow 0} \text{vol}(S^{2r-1}(v)) = 0$$

and  $S$  is a strong deformation retraction of  $\Xi$ , there is a smooth form  $\check{\phi}$  on  $X$  such that

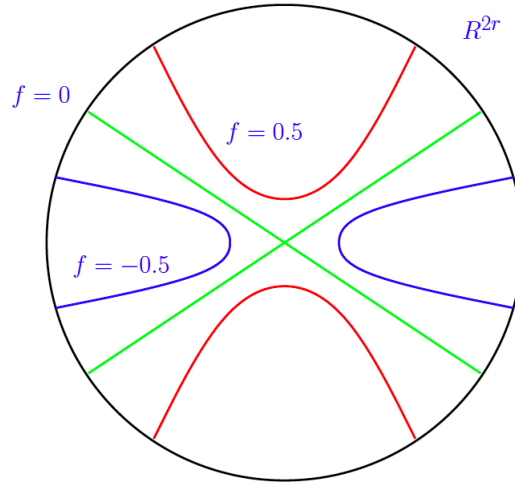
$$\check{\phi}|_{\Xi-o} - \Phi = d\check{\psi}$$

for some smooth  $(2r-2)$ -form  $\check{\psi}$  on  $\Xi - o$ . Now, away from  $S$ , glue  $\check{\phi}$  and  $\Phi$  together to a smooth form  $\hat{\Phi}$  on  $X - o$ , and meanwhile extend  $\check{g}$  to  $\hat{g}$  making  $\hat{\Phi}$  a calibration on  $X - o$ .

By Remark 2.13,  $[[S]]$  is homologically mass-minimizing. However, it is impossible to calibrate  $S$  using a smooth calibration  $\bar{\Phi}$  on  $(X, \hat{g})$ . Since if it were the case, according to Remark 4.5 the tangent cone of  $S$  at  $o$ , a Simons cone, would be calibrated in  $(T_o X, \bar{\Phi}_o, \hat{g}_o)$ . But  $\bar{\Phi}_o$  can calibrate certain hyperplanes only. Contradiction!

Now we give a concrete construction for such  $S$ .

**Example 2:** Let  $T$  be an oriented compact  $(2r - 1)$ -dimensional smooth manifold. One can embed  $K = S^{r-1} \times D^r$  into some small ball on it. After surgery along  $S^{r-1} \times S^{r-1}$ , one gets a manifold  $T'$ . The oriented  $2r$ -dimensional smooth manifold  $W$  obtained by the union of



$[-0.5, 0.5] \times (T - K)$  and the region between  $\{f = -0.5\}$  and  $\{f = 0.5\}$  in the picture under the identification diffeomorphisms of  $\{t\} \times S^{r-1} \times S^{r-1}$  with  $\{f = t\} \cap S^{2r-1}(1)$  is a cobordism between  $T$  and  $T'$  (corresponding to  $t = -0.5$  and  $t = 0.5$  respectively). Here  $f$  is defined on  $B^{2r}(1) \subset \mathbb{R}^r \times \mathbb{R}^r$  by  $f(\vec{x}, \vec{y}) = -\|\vec{x}\|^2 + \|\vec{y}\|^2$ , and  $f^{-1}(0)$  is the truncated Simons cone.

Take two copies of  $W$ . Glue the same boundaries. Then one gets an orientable compact  $2r$ -dimensional manifold  $X$ . Now extend the Euclidean metric on the region between  $\{f = -0.25\}$  and  $\{f = 0.25\}$  in the first copy to a metric on  $X$ . Let  $(S, o)$  be the singular hypersurface in the first  $W$  corresponding to  $t = 0$ . Apparently  $[S] \neq [0]$  in  $H_{2r-1}(X; \mathbb{R})$  (by intersection with a “ $t$ -circle”). Then Example 1 shows that  $S$  can be calibrated by a coflat calibration  $\Phi$  singular only at  $o$  with respect to some metric  $g$  on  $X$ .

**Remark 4.12.** By cross-products examples with more complicated singularity can be generated. For instance, let  $S_i, \Phi_i, X_i, g_i$  be given above for  $i = 1, 2$ . Then  $S_1 \times S_2$  with singularity  $S_1 \vee S_2$  is calibrated by the coflat calibration  $\Phi_1 \wedge \Phi_2$  with singular set  $S_1 \vee S_2$  in the cartesian product  $(X_1, g_1) \times (X_2, g_2)$ .

**Remark 4.13.** Suppose  $C$  is a cone of higher codimension  $\mathbb{R}^n$  that has a calibration singular at most at one point. Consider  $\Sigma_C \triangleq (C \times \mathbb{R}) \cap S^n(1)$  in  $\mathbb{R}^{n+1}$ . Choose an  $n$ -dimensional oriented compact manifold  $T$  with nontrivial  $H_k(T; \mathbb{R})$ . Take an embedded oriented connected compact submanifold  $M$  that represents a nonzero class of  $H_k(T; \mathbb{R})$ . In smooth disks around a point of

$M$  and a smooth point of  $\Sigma_C$  respectively one can simultaneously connect  $T$  and  $S^n(1)$ ,  $M$  and  $\Sigma_C$  through one surgery along  $S^0 \times S^n$  (i.e., connected sum). Denote by  $X$  and  $S$  the obtained manifold and submanifold (singular at two points). Then  $[S] \neq 0 \in H_k(X; \mathbb{R})$  and similarly there exists a global calibration pair of  $S$  by the proof of Theorem 4.6.

**Example 3:** Let  $M$  be the smooth “fiber” corresponding to  $\{t = -0.3\}$  in the first copy of  $W$  in Example 2. Note that  $\Phi$  is already a coflat calibration of  $S$  on  $(X, g)$ . According to §3.3, §3.4 and Remark 3.8, one can modify the calibration to  $\tilde{\Phi}$  and conformally change  $g$  to  $\tilde{g}$  in a neighborhood of  $M$  away from  $S$  such that  $\tilde{\Phi}$  becomes a coflat calibration calibrating both  $S$  and  $M$  in  $(X, \tilde{g})$ .

However the homologically mass-minimizing submanifold  $M$  cannot be calibrated by any smooth calibration in  $(X, \tilde{g})$ . If it were, then  $S$  must be calibrated by the same smooth calibration as well which would lead to a contradiction. This implies that all coflat calibrations of  $M$  in  $(X, \tilde{g})$  share at least a common singular point. For such creatures of higher codimension, one can consider  $M \times \{\text{a point}\}$  in the Riemannian product of  $(X, \tilde{g})$  and a compact oriented manifold. See Remark 4.12 for more complicated examples.

Next we consider the non-orientable case.

**Example 4:** Based upon  $C_{3,4}$  one can get an eight-dimensional oriented compact connected submanifold  $S$  with one singular point in some oriented manifold  $X^9$  with  $[S] \neq [0] \in H_8(X; \mathbb{R})$  by the method of Example 2. Now blow up at a regular point of  $S$ . Call the resulting manifold and submanifold  $\check{X}$  and  $\check{S}$  respectively.

By the Seifert-van Kampen theorem  $\pi_1(\check{X}) \cong \pi_1(X) * \pi_1(\mathbb{R}P^8)$ . The isomorphism of  $\pi_1(\mathbb{R}P^8) \cong \mathbb{Z}_2$  trivially extends to a homomorphism  $\pi_1(\check{X}) \rightarrow \mathbb{Z}_2$ , which canonically determines a two-sheeted cover  $\bar{X}$  of  $\check{X}$ . Denote the lifting of  $\check{S}$  by  $\bar{S}$ . Note that  $\bar{X} \cong X \# X$  and  $\bar{S} \cong S \# S^{\text{opposite orientation}}$ . By Mayer-Vietoris sequences, one has

$$H_8(\bar{X}; \mathbb{R}) \cong H_8(X; \mathbb{R}) \oplus H_8(X; \mathbb{R}), \text{ and}$$

$$[\bar{S}] = [(S, -S)] \neq [0] \text{ in } H_8(\bar{X}; \mathbb{R}).$$

Now create a  $\mathbb{Z}_2$ -invariant metric  $\bar{g}$  on  $\bar{X}$  such that the orientable  $\bar{S}$  can be calibrated (by a twisted calibration in the sense of [Mur91]).

$\check{S}$  induces a  $d$ -closed integral current mod 2,  $[[\check{S}]]_2$  (see [Zie62]), representing a non-zero  $\mathbb{Z}_2$ -homology class  $[\check{S}]_2$ . We want to show that  $[[\check{S}]]_2$  is  $\mathbf{M}^2$ -minimizing in  $[\check{S}]_2$  under the induced metric  $\check{g}$  on  $\check{X}$ , where the **mass**  $\mathbf{M}^2(\cdot)$  of an integral current mod 2 is the infimum of the mass of all integral representatives.

Suppose  $K - [[\check{S}]]_2 = dW$  in the sense of mod 2 for an integral current  $K$  of finite mass and  $W$  a top dimensional integral current mod 2. Then the lifting expression to  $\bar{X}$  becomes  $\bar{K} - [[\bar{S}]] = d\bar{W}$  in the sense of mod 2. (Since  $\bar{S}$  is orientable,  $[[\bar{S}]]$  is an integral current up to a choice of orientation.) Now  $\bar{X}$  is oriented and  $\bar{W}$  is of top dimension, so  $\bar{W}$  comes from the quotient of  $\tilde{W}$  by 2 where  $\tilde{W}$  is the integral current with multiplicity one on  $\mathbf{spt}(\tilde{W})$  and orientation inherited from  $\bar{X}$ . Restrict  $\tilde{W}$  to the connected component of  $\mathbf{spt}(\tilde{W})$  to  $\bar{S}$  and denote it by  $\tilde{W}^\circ$ . Assign  $[[\bar{S}]]$  the orientation induced from  $\tilde{W}^\circ$ . Let  $-\bar{K}^\circ \triangleq d\tilde{W}^\circ - [[\bar{S}]]$ . It follows

$$\mathbf{M}_{\bar{g}}(\check{S}) = \frac{1}{2} \mathbf{M}_{\bar{g}}([[\bar{S}]]) \leq \frac{1}{2} \mathbf{M}_{\bar{g}}(\bar{K}^\circ) \leq \mathbf{M}_{\bar{g}}(K).$$

Running  $K$  through all the integral representatives of  $[[\check{S}]]_2$  one has

$$\mathbf{M}_g^2(\check{S}) = \mathbf{M}_g^2([[\check{S}]]_2).$$

Let  $K_2$  be the integral current mod 2 of an integral current  $K$  with  $[K_2] = [[\check{S}]]_2$ . Then

$$\mathbf{M}_g^2([[\check{S}]]_2) \leq \mathbf{M}_g^2(K_2).$$

Namely,  $[[\check{S}]]_2$  is  $\mathbf{M}_g^2$ -minimizing in its  $\mathbb{Z}_2$  homology class.

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*Current address: School of Mathematics and Statistics, Northeast Normal University, 5268 RenMin Street, Nan-Guan District, ChangChun, JiLin 130024, P.R. China*  
*E-mail address: yongsheng.chang@gmail.com*